

# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1275

THE SOLUTION OF THE LAMINAR-BOUNDARY-LAYER EQUATION FOR  
THE FLAT PLATE FOR VELOCITY AND TEMPERATURE FIELDS  
FOR VARIABLE PHYSICAL PROPERTIES AND FOR THE  
DIFFUSION FIELD AT HIGH CONCENTRATION

By H. Schuh

Translation of ZWB Forschungsbericht Nr. 1980, August 1944



Washington

May 1950



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SUMMARY

In connection with Pohlhausen's solution for the temperature field on the flat plate, a series of formulas were indicated by means of which the velocity and temperature field for variable physical characteristics can be computed by an integral equation and an iteration method based on it. With it, the following cases were solved: On the assumption that the viscosity simply varies with the temperature while the other fluid properties remain constant, the velocity and temperature field on the heated and cooled plate, respectively, was computed at the Prandtl numbers 12.5 and 100 (viscous fluids). A closer study of these two cases resulted in general relations: The calculations for a gas of Pr number 0.7 (air) were conducted on the assumption that all fluid properties vary with the temperature, and the velocities are low enough for the heat of friction to be discounted. The result was a thickening of the boundary layers, but no appreciable modification in shearing stress or heat-transfer coefficient. The effects of density and viscosity or density and heat conductivity have opposite effect for velocity and temperature field and almost cancel one another. Formulas allowing for the heat produced by the friction were indicated, but no calculations were carried through in view of the already existing report by Crocco. The methods of solution developed here were finally applied also to the case of diffusion of admixtures, where at higher concentration finite transverse velocities occur at the wall.

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\*"Über die Lösung der laminaren Grenzschichtgleichung an der ebenen Platte für Geschwindigkeits- und Temperaturfeld bei veränderlichen Stoffwerten und für das Diffusionsfeld bei höheren Konzentrationen." Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB) Berlin-Adlershof, Forschungsbericht Nr. 1980, August 18, 1944.

## I. INTRODUCTION

The laminar-boundary-layer equation for the flat plate lined up with the flow and with constant fluid properties was solved by Prandtl (reference 1) and Blasius (reference 2) for the flow field and by Pohlhausen (reference 3) for the temperature field. Since temperature and velocity field coincide when kinematic viscosity ( $\nu$ ) and temperature conductivity ( $\alpha$ ) are identically equal ( $Pr = \frac{\nu}{\alpha} = 1$ ), Pohlhausen's formula for the temperature field contains a solution for the velocity field also in the form of an integral equation. Piercy and Preston (reference 4), proceeding from a rough approximation, indicated that, with the aid of this integral equation and an iteration method, the well-known Blasius solution can be obtained in a few steps. This method of solution has the advantage of being simple and requiring relatively little time. It is - as is shown in the following - particularly suitable for boundary-layer calculations involving variable fluid properties, because a first, and usually fairly close, approximation, is already available in the solution for constant fluid properties.

Crocco (reference 5) and von Kármán and Tsien (reference 6) (see also reference 9, 10) computed velocity and temperature field for variable fluid properties. In both reports, the differential equations are put in a different form from the elsewhere conventional boundary-layer calculation by changing to new variables. Crocco obtains two simultaneous differential equations of the second order which he solves for a gas with the Prandtl number  $Pr = 0.725$  (air). Von Kármán and Tsien treat the case of  $Pr = 1$  and have to solve only one differential equation, since then the temperature is related in a simple manner to the velocity.

In the following, it is shown that a number of boundary-layer problems for the flat plate can be solved in a comparatively simple manner, involving merely quadrature, by means of the cited integral equation and an iteration method.

## II. SOLUTION OF BOUNDARY-LAYER EQUATION FOR VARIABLE PHYSICAL PROPERTIES

The boundary-layer equations for velocity and temperature field at the flat plate at variable density read (reference 3)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \quad (1)$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad (1a)$$

$$c_p \rho u \frac{\partial T}{\partial x} + c_p \rho v \frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) \quad (2)$$

with  $u$  and  $v$  the speed in flow direction and at right angle to it,  $T$ , the temperature,  $x$  the distance from the plate leading edge,  $y$  the distance from the wall,  $\rho$  the density,  $\mu$  the viscosity,  $c_p$  the specific heat, and  $\lambda$  the heat conductivity. In the equation for the temperature field, the heat produced by friction is, at first, not taken into account; as long as the speeds are not excessive and the temperature differences not too small, this is justified.

For constant density, equations (1) and (2) can be reduced to an ordinary differential equation (reference 3) on the assumption that  $u$  and  $T$  are a function merely of the one (dimensionless)

coordinate  $\xi = \frac{y}{2} \sqrt{\frac{U}{\nu x}}$ . Since the density depends only on the temperature, the idea suggests itself that the same simplification is possible also for variable density. We put

$$\frac{\mu}{U} = \omega(\xi) \quad \frac{T - T_0}{T_1 - T_0} = \theta(\xi) \quad \xi = \frac{y}{2} \sqrt{\frac{U}{\nu_k x}} \quad (3)$$

where  $U$  is the velocity at the edge of the boundary layer,  $T_0$  and  $T_1$  the wall temperature and the temperature at the edge of the boundary layer, respectively. The quantity  $\nu_k$  in the dimensionless  $\xi$  denotes the kinematic viscosity for the fixed temperature  $T_k$ , for which in suitable manner the wall temperature ( $k = 0$ ), or the temperature at the edge of the boundary layer ( $k = 1$ ), is chosen. The boundary conditions for flow and temperature field read

$$\begin{array}{llll} y = 0 & \xi = 0 & \omega = 0 & \theta = 0 \\ y \rightarrow \infty & \xi \rightarrow \infty & \omega = 1 & \theta = 1 \end{array} \quad (4)$$

Putting

$$\frac{\rho}{\rho_k} = \psi(\xi) \quad \frac{\mu}{\mu_k} = \varphi(\xi) \quad \frac{\lambda}{\lambda_k} = \chi(\xi) \quad (5)$$

where the subscript  $k$  denotes the density at temperature  $T_k$ , gives by (1a)

$$\rho v = \sqrt{\frac{\nu_k}{Ux}} \left( \rho u \xi - \int_0^\xi \rho u \, d\xi \right) \quad (6)$$

hence, by (1), after introduction of (5) and (6), the more suitable form

$$\frac{d}{d\xi} \left( \phi \frac{d\omega}{d\xi} \right) = - \left( \phi \frac{d\omega}{d\xi} \right) \frac{f}{\phi} \quad f = 2 \int_0^\xi \psi \omega \, d\xi \quad (7)$$

From (7), regarded as differential equation for the quantity  $\phi \frac{d\omega}{d\xi}$  and  $f$  temporarily as a known function of  $\xi$ , the following expression for  $\omega$  is derived

$$\omega = \frac{J(\xi)}{J(\infty)} ; \quad J(\xi) = \int_0^\xi \frac{1}{\phi} e^{-\int_0^\xi \frac{f}{\phi} \, d\xi} \, d\xi \quad (8)$$

This disposes of the integration constant from consideration of the boundary condition (4). Likewise, there is afforded for the dimensionless temperature  $\theta$  the expression

$$\theta = \frac{K(\xi)}{K(\infty)} \quad K(\xi) = \int_0^\xi \frac{1}{\chi} e^{-Pr_k \int_0^\xi \frac{f}{\chi} \, d\xi} \, d\xi \quad (9)$$

where  $Pr_k$  is the Prandtl number with the density at temperature  $T_k$ .

For constant density ( $\phi = \psi = \chi = 1$ ), velocity and temperature field are independent of each other and (9) gives the Pohlhausen expression (reference 3) for the temperature field, which represents the solution for the velocity field at  $Pr = 1$  ( $\nu = a$ ). When the velocity field is known, the solution for the temperature by (9) is obtainable by simple quadrature. But the calculation of the velocity field runs into difficulties, at first, because in (8) the still unknown velocity appears on the right-hand side in the expression for  $f$ . The methods of solution by Piercy and Preston proceed from a random approximation for  $\omega$  with which  $f$  and  $J(\xi)$  in (8) are

computed. The improved value  $\omega$  obtained forms the starting point for the next step, etc. Figure 1 represents the several steps of this approximation method. The intentionally rough approximation  $\omega = 1$  over the entire boundary layer was chosen as original solution; the corresponding first approximation<sup>1</sup>  $\omega$  is given by the error integral. After the third approximation, the shearing stress shows a mere difference of 4.5 percent from the exact value. Instead of continuing the process mechanically, the final solution to be expected was estimated from the variation of the previously computed approximations and utilized as basis for the subsequent step; the solution  $\omega$  contained but a  $\frac{1}{2}$ -percent error in shearing stress.

With this method of solution, the improvement effected by each step can be estimated according to order of magnitude. The equations (8) and (9) are identical for constant density and  $Pr = 1$ . Assuming that the approximate solution for  $\omega$  was such that for each individual value  $\omega$  the corresponding  $\xi$  coordinate differed by a constant factor  $\chi$  from the  $\xi$  coordinate of the exact solution, the effect of factor  $\chi$  is then obviously just as great as that of quantity  $Pr$  for the temperature field. Pohlhausen found, on the basis of his numerical calculations, that the heat-transfer coefficient is proportional to  $\sqrt[3]{Pr}$ , thus the shearing stress at the wall is afflicted at each new step by an error of only about one-third of the error of the preceding step.

For variable density, the discussed solution steps of "mathematical nature" can be combined with the steps of "physical nature":

Step 1: as starting point the known solutions for constant density are assumed:

- (a) The Blasius solution (reference 2) for the velocity profile
- (b) Pohlhausen's method for the temperature field

Step 2:

- (a) Calculation of velocity profile by (8), the temperature variation being based on the density of the temperature profile according to step 1(b)
- (b) Calculation of temperature field by (9) with the velocity profile according to step 2(a); relation of density to temperature as in step 2(a)

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<sup>1</sup>It took a subsidiary worker 10 hours to reach the final solution of the velocity field in figure 1.

The process is repeated till the final solution is sufficiently exact, usually requiring three to four steps.

A few general remarks about the influence of the temperature variability of the physical properties.- The flow with constant physical properties can be regarded as first approximation, and the problem is then to ascertain the differences which are produced by variable physical properties. The quality of this approximation depends, of course, on the temperature assumed for the physical properties at the "isothermal" flow. Choosing the wall temperature or the temperature at the edge of the boundary layer as reference temperature for the isothermal flow so results on the basis of physical point of view as well as on the basis of the equations that an increase of the viscosity or density inside the boundary layer is accompanied by an increase in the resistance; similarly, an increase in heat conductivity and density effects a greater heat transfer. But the magnitude of the effect of variability of the separate physical properties is contingent upon the ratio of the boundary-layer thickness of the temperature and velocity field. (The ratio of both is proportional according to Pohlhausen.)

This is illustrated by the following case, which is, at the same time, of practical importance. The temperature boundary layer is assumed very small compared to the flow boundary layer; consequently, the variation of the physical properties within the thermal boundary layer can be disregarded for the shearing stress and the latter computed as if the temperature at the edge of the boundary layer reaches to the wall. The same holds for the velocity profile, with the exception of a small area within the thermal boundary layer, where the velocity profile by the viscosity variation is deformed correspondingly. But for the temperature profile this area is exactly decisive.

From the equality for the shearing stresses the velocity gradients at the walls are:

$$\left( \frac{\partial u}{\partial y} \right)_0 = \frac{\mu_1}{\mu_0} \left[ \left( \frac{\partial u}{\partial y} \right)_0 \right]_{11} \quad \text{for } Pr' \rightarrow \infty \quad (10)$$

the subscript 11 denotes the "isothermal" flow with the physical properties at temperature  $T_1$ . The variability of density is noneffective for the field of flow, in this instance. It can be mathematically derived from the formulas (8) and (9). The ratios for the temperature field are discussed in the next chapter by means of the two examples.



## III. FLOW AND TEMPERATURE FIELD FOR VISCOUS FLUIDS

In accordance with the physical properties of viscous fluids, the velocity and temperature field were computed on the assumption that only the viscosity should change with the temperature by the following formula

$$\frac{\mu}{\mu_k} = \left( \frac{T_k + T_c}{T + T_c} \right)^b \quad (11)$$

where  $b$  and  $T_c$  are constants, chosen so as to reproduce the temperature variation as closely as possible. The subscript  $k$  is to be 0 or 1, depending upon the choice of the physical properties in the dimensionless  $\xi$ . The choice was  $b = 3$  (viscous lubricating oil) and the two cases of a heated and cooled plate computed with

$$\frac{\mu_0}{\mu_1} = \frac{1}{8} \quad \text{and} \quad 8 \quad \text{and} \quad Pr_0 = 12.5 \quad \text{and} \quad 100; \text{ it thus concerned identically}$$

great temperature differences of the same fluid, since  $Pr_0$  is

for the present formed with the physical properties at wall temperature. Choosing  $T_0$  as reference temperature gives by (11)

$$\varphi = \frac{\mu}{\mu_0} = \left( \frac{1}{\theta \left( \sqrt[3]{\frac{\mu_0}{\mu_1}} - 1 \right) + 1} \right)^3 \quad (12)$$

The result of the calculation by the iteration method of the preceding section is seen in figures 2 and 3. In both graphs, the dimensionless wall distances  $\xi_0$  and  $\xi_1$ , formed with  $\mu_0$  and  $\mu_1$ , are plotted to

the scale  $1:\sqrt{8}$  and  $\sqrt{8}:1$ , respectively, so that the actual wall distance  $y$  is the same for both abscissas. Besides the solution  $\omega$ , which took three steps to compute, the isothermal velocity profiles  $(\omega)_0$  and  $(\omega)_1$  at constant density at temperature  $T_0$  and  $T_1$  are shown plotted against the dimensionless coordinates  $\xi_0$  and  $\xi_1$ .<sup>2</sup>

In the subsequent compilation  $\tau_0$  and  $\alpha$  denote the shearing stress at the wall and the heat-transfer coefficient  $\left( \alpha = \lambda \left( \frac{\partial \theta}{\partial y} \right)_0 \right)$ ;

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<sup>2</sup>For the "isothermal" temperature profiles  $(\theta)_0$  and  $(\theta)_1$ , the Prandtl numbers at temperatures  $T_0$  and  $T_1$  must be inserted. For example, in figure 2:  $Pr_0 = 12.5$  and  $Pr_1 = 100$ .

$(\tau_o)_1$  and  $(\alpha)_o$  are the corresponding values in isothermal flow with the viscosity at temperature  $T_1$  and  $T_o$ .

Table I

	$\frac{\tau_o}{(\tau_o)_1}$	$\frac{\alpha}{(\alpha)_o}$	$Pr_o$	$\left(\frac{d\omega}{d\xi_o}\right)_o$	$\left(\frac{d\theta}{d\xi_o}\right)_o$
Heated wall	0.841	1.20	12.5	1.58	1.84
Cooled wall	1.08	0.98	100	0.255	3.01

Although in both cases the thermal boundary-layer thickness is far from small compared to the flow boundary layer, the shearing stress can still be computed satisfactorily by the isothermal formula with the viscosity of the wall temperature. The assumptions to equation (10) are thus shown for  $Pr > 10$ .

The conditions are more complicated for the heat-transfer coefficient; from (9), it follows that the heat-transfer coefficient

$\alpha$  is proportional to  $\sigma(Pr) \sqrt{\frac{U}{\nu x}}$ , wherein, according to Pohlhausen,

$\sigma$  is, with high accuracy, assumed as  $0.664 \sqrt[3]{Pr}$ . Bearing in mind that  $Pr = \frac{\nu}{\alpha}$ , it follows that the heat-transfer coefficient is inversely

proportional to the sixth root of the viscosity. Since all physical properties except the viscosity have been assumed constant, there results, when it is referred once to the wall temperature, the other time to the temperature at the edge of the boundary layer

$$\frac{(\alpha)_o}{(\alpha)_1} = \left(\frac{\mu_1}{\mu_o}\right)^{1/6} \quad (13)$$

A comparison with the foregoing tabulation indicates that  $(\alpha)_1$  supplies a poorer approximation for the heat-transfer coefficient than  $(\alpha)_o$ ; this is readily explained by the variation of the velocity profile (figures 2 and 3). It is to be expected that the conditions are similar at higher Prandtl numbers.

Another reference point for the heat-transfer coefficient is found in the velocity gradient at the wall; by (10) and allowing for (3), there follows

$$\frac{\left(\frac{\partial u}{\partial y}\right)_o}{\left[\left(\frac{\partial u}{\partial y}\right)_o\right]_{11}} = \frac{\mu_1}{\mu_o} \quad \frac{\left(\frac{\partial u}{\partial y}\right)_o}{\left[\left(\frac{\partial u}{\partial y}\right)_o\right]_{1o}} = \sqrt{\frac{\mu_1}{\mu_o}} \quad (14)$$

the subscripts 1o and 11 denoting isothermal flow at temperature  $T_o$  and  $T_1$ . These relations are confirmed in figures 2 and 3.

With these formulas, limits can be indicated for the heat-transfer coefficients (figures 2 and 3). One is given according to (13) by  $(\alpha)_1$ , because the velocity profile  $(w)_1$  yields at all points higher velocities at cooled and lower velocities at heated wall. The other limit is given by a velocity profile of isothermal form, where the abscissa scale is so modified that its gradient at the wall agrees with the actual velocity distribution. From the remark about the convergence of the method of solution in II, it follows then that the heat-transfer coefficient is proportional to the third root of the velocity gradient at the wall; for this extreme value, the second

equation of (14) gives:  $\sqrt[6]{\frac{\mu_1}{\mu_o}} (\alpha)_o$ . Summed up, the limits of the heat-transfer coefficients, by a change in viscosity, are

$$\sqrt[6]{\frac{\mu_o}{\mu_1}} (\alpha)_o \leq \alpha \leq \sqrt[6]{\frac{\mu_1}{\mu_o}} (\alpha)_o \quad (15)$$

the upper signs applying to heated, the lower to cooled wall.

Hence, the following approximate rule for viscous fluids ( $Pr > 10$ ): For computing the resistance, the physical properties are referred to the temperature at the edge of the boundary layer; for heat transfer, to the wall temperature.

#### IV. FLOW AND TEMPERATURE FIELD AT $Pr = 0.7$ (AIR)

##### FOR TEMPERATURE VARIABILITY OF EVERY PHYSICAL PROPERTY

In the  $-50^\circ$  to  $140^\circ$  temperature range the physical properties of the air can be represented by the following formulas



The explanation for it is that in air the growth of the viscosity with the temperature acts in the sense of a resistance increase, the drop in density in the sense of a resistance decrease, and both effects practically cancel one another at  $Pr = 0.7$ , where thermal and flow boundary-layer thickness are about equally great. The conditions for the temperature field are almost identical, because the heat conductivity and the viscosity are similarly affected by the temperature.

The frictional heat can be allowed for in similar manner; equation (2) contains then an additive term  $\mu \left( \frac{\partial u}{\partial y} \right)^2$  on the right-hand side, and the solution reads

$$\left. \begin{aligned} \theta &= \frac{1 + B(\infty)}{A(\infty)} A(\xi) - B(\xi) \\ A(\xi) &= \int_0^\xi \frac{1}{\chi} e^{-R(\xi)} d\xi \\ B(\xi) &= 2 \frac{Pr_k \Delta T_e}{(T_1 - T_0)} \int_0^\xi \frac{1}{\chi} \left[ \int_0^\xi \phi \omega'^2 e^{R(\xi)} d\xi \right] e^{-R(\xi)} d\xi \\ R(\xi) &= Pr_k \int_0^\xi \frac{f(\xi)}{\chi} d\xi \quad \Delta T_e = \frac{U^2}{2c_p} \end{aligned} \right\} \quad (16)$$

The iteration method can be applied again, although a little more paper work is involved. For constant physical properties, equation (16) reduces to Eckert's solution (reference 8). The thermometer problem (vanishing temperature gradient at the wall) can also be solved by suitable variation of the integration constant. In view of Crocco's calculations for a gas with  $Pr = 0.725$ , it was decided not to calculate any model problems by the new method.

## V. APPLICATION TO A DIFFUSION PROBLEM

The concentration field for the problem of diffusion at the flat plate can be calculated in the same manner as the temperature field.<sup>3,4</sup> The differential equation reads

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = k \frac{\partial^2 c}{\partial y^2} \quad (17)$$

where  $k$  is the diffusion factor and  $c$  the concentration which is defined as quantity of gas or vapor per unit volume. The physical properties are regarded as constant, but it is also taken into account that for greater concentrations the velocity  $v$  at the wall no longer disappears, as already pointed out by Nusselt (reference 7). When fluid from a wall is vaporized, say by a gas such as gas flowing along a wetted wall, substance passes continuously into the flow. Hence  $v(0) > 0$  at the wall. When, on the other hand, vapor condenses at the wall or when air containing ammonia, for example, passes over blotting paper impregnated with hydrochloric acid, it results in  $v(0) < 0$ . The boundary conditions for  $v$  are according to the equations (100) and (101) of reference (7):

$$- \frac{k \left( \frac{\partial c}{\partial y} \right)_0}{c_0 \left( \frac{p}{p_0} - 1 \right)} = v(0) \quad (18)$$

where  $c$  is the concentration of the gas or vapor, for which the wall is permeable,  $c_0$  the concentration at the wall,  $p_0$  the corresponding partial pressure, and  $p$  the total pressure.

Introduction of the flow velocity  $U$  and the dimensionless  $\xi$  results in

$$\left( \frac{v}{U} \right)_0 = - \frac{1}{2} \frac{k}{c_0} \frac{c_1 - c_0}{\left( \frac{p}{p_0} - 1 \right) U} \left( \frac{dc}{d\xi} \right)_0 \sqrt{\frac{U}{\nu x}} \quad c = \frac{c - c_0}{c_1 - c_0} \quad (19)$$

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<sup>3</sup>Eckert reported a solution of this problem at the 1943 meeting of the VDI committee for heat research in Bayreuth, where an approximation method similar to Pohlhausen's method for the flow boundary layer was used.

<sup>4</sup>"Damkohler's estimate for the present problem was published in Z. für Elektrochemie, 1942, p. 178.

where  $c_0$  and  $c_1$  are the concentration at the wall and at the edge of the boundary layer. Similarly to (5), it gives

$$\frac{v}{U} = \sqrt{\frac{v}{Ux}} \left[ \omega \xi - \int_0^\xi \omega d\xi + \frac{M}{2} \right] \quad M = -\frac{k}{v} \frac{c_1 - c_0}{c_0 \left( \frac{p}{p_0} - 1 \right)} \left( \frac{dc}{d\xi} \right)_0 \quad (20)$$

and similarly to (9), the solution for the concentration field

$$\left. \begin{aligned} C &= \frac{L(\xi)}{L(\infty)} & L(\xi) &= \int_0^\xi e^{-\frac{v}{k} \int_0^\xi (f(\xi) - M) d\xi} d\xi \\ f(\xi) &= 2 \int_0^\xi \frac{u}{U} d\xi \end{aligned} \right\} \quad (21)$$

where  $\frac{v}{k}$  is a quantity analogous to the Prandtl number. To obtain the velocity  $\omega$ , simply put  $\frac{v}{k} = 1$  in equation (21).

The concentration gradient at the wall is contained in  $M$ ; but (21) can, in the first instance, be solved for any  $M$  values and the quantity

$$N = -\frac{k}{v} \left( \frac{c_1 - c_0}{c_0 \left( \frac{p}{p_0} - 1 \right)} \right) \text{ computed with the aid of the value obtained from}$$

the solution for  $\left( \frac{dc}{d\xi} \right)_0$ . The velocity and concentration fields for the

calculated  $M$  and  $N$  values are represented in figures 5 and 6, the concentration gradient at the wall, in figure 7.  $M > 0$  denotes evaporation at the plate;  $M < 0$ , condensation and absorption at the plate; the

value 0.6 chosen for the quantity  $\frac{v}{k}$  is applicable in good approxima-

tion for the diffusion of water and ammonia in air.<sup>5</sup> Strictly speaking, for the specified higher concentrations, the density and viscosity of the two fluids are dependent on the concentration; and the diffusion factor, on the temperature. Cases of that kind can be calculated with the aid of the described method. If the diffusion is

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<sup>5</sup>According to Ten Bosch: Die Wärmeübertragung, Berlin 1936, pp. 189 and 257.

accompanied by a heat transfer, the solution for the concentration field can equally be applied to the temperature field with good approximation. In the same way, the heat transfer can be derived from the solution for the concentration field, when air is exhausted or blown at the plate with transverse velocities at the wall corresponding to equation (20).

#### CONCLUDING NOTE

After completion of the calculations the writer received knowledge of a report by Schlichting and Bussmann (reference 11) about the velocity profile at the flat plate for exhaustion where the transverse velocity at the wall was expressed by

$$v_o(x) = -\frac{C}{2} \sqrt{\frac{WU_o}{x}}$$

Between the present value  $M$  and  $C$  the following relation exists (see also (19) and (20)).

$$C = -M$$

The present velocity distributions agree to about 1 percent with those calculated by Schlichting (by a different method), with exception of  $M = 1$ , where the writer plainly chose too few approximation steps and the differences are therefore a little greater. For the present calculation three steps were usually sufficient.

Translated by J. Vanier  
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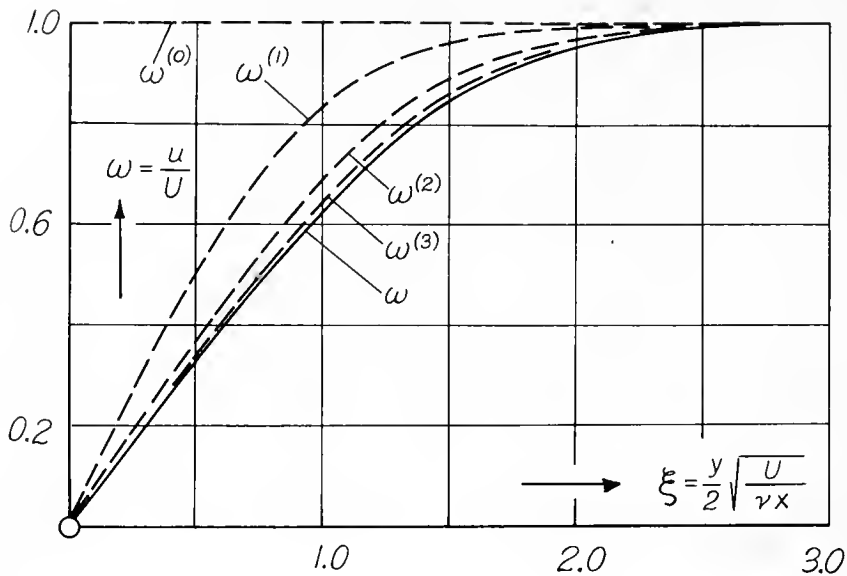


Figure 1.- The several approximations for computing the velocity distribution at the flat plate by the method of Preston and Piercy (constant physical quantities).

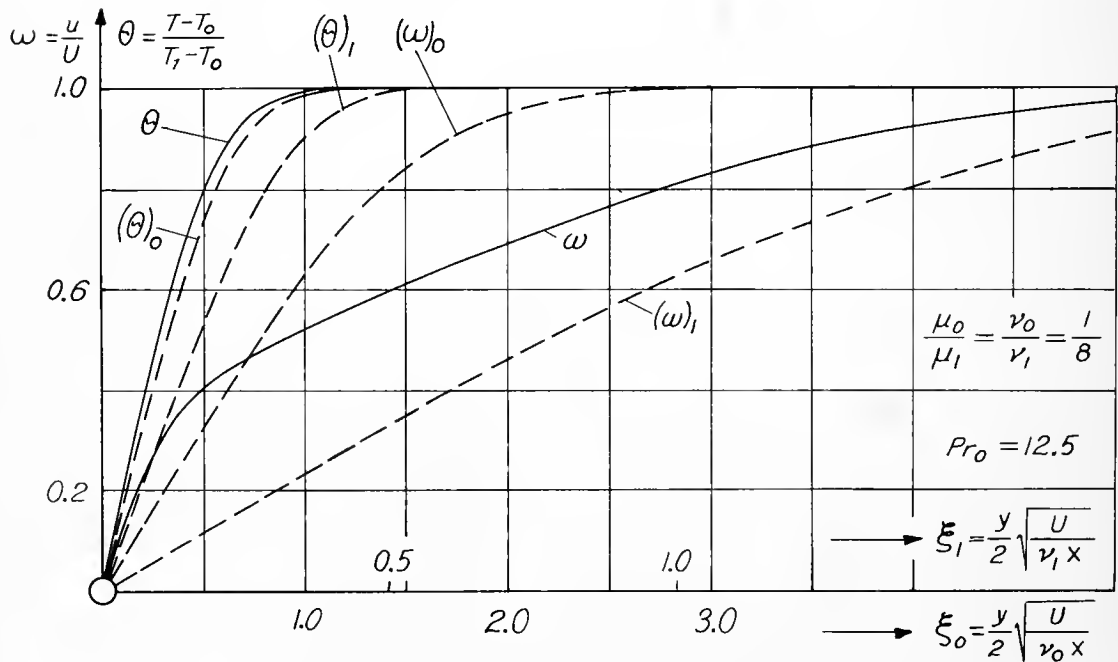


Figure 2.- Velocity and temperature distribution at a heated plate for variable viscosity. Viscosity exponent  $b = 3$ .  $(\omega)_0, (\theta)_0$ , and  $(\omega)_1, (\theta)_1$  isothermal velocity and temperature distributions,  $\nu_0$  and  $\nu_1$  kinematic viscosity at wall temperature  $T_0$  and temperature  $T_1$  at the edge of the boundary layer.

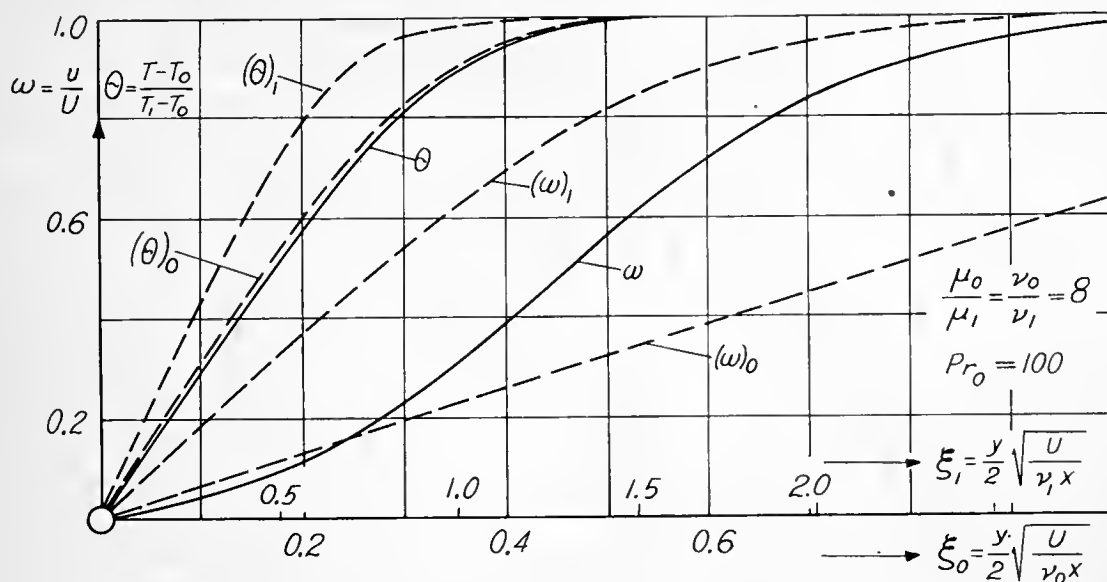


Figure 3.- Velocity and temperature distribution at the cooled plate. Viscosity exponent  $b = 3$ .  $(\omega)_0$ ,  $(\theta)_0$ , and  $(\omega)_1$ ,  $(\theta)_1$  isothermal velocity and temperature distributions,  $\nu_0$  and  $\nu_1$  kinematic viscosity at wall temperature  $T_0$  and temperature  $T_1$  at the edge of the boundary layer.

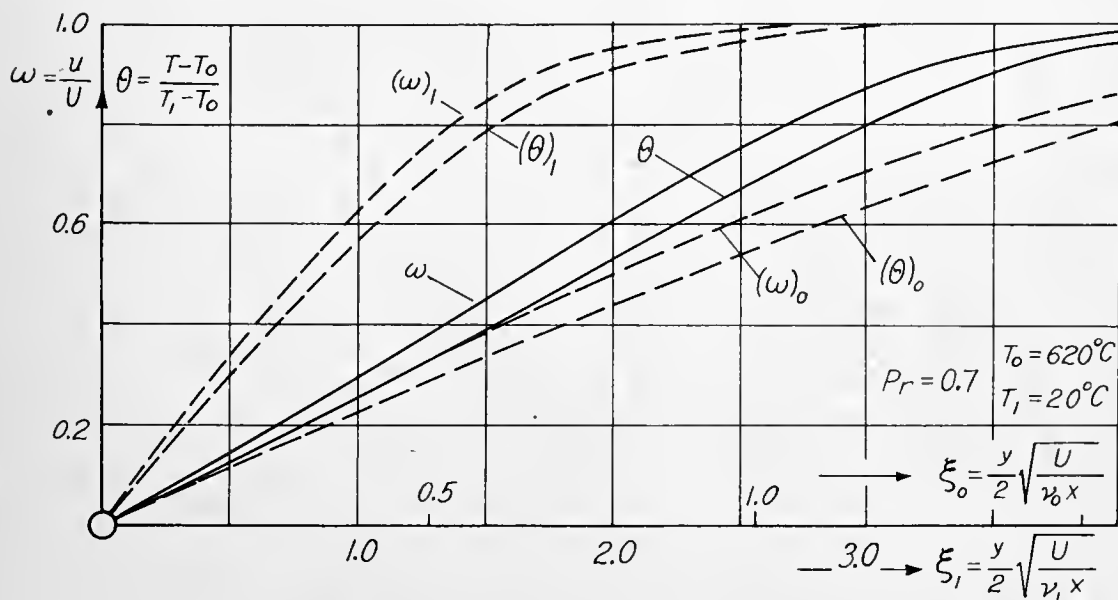


Figure 4.- Velocity and temperature distribution at a heated plate for  $Pr = 0.7$  (air); all physical quantities constant with temperature.

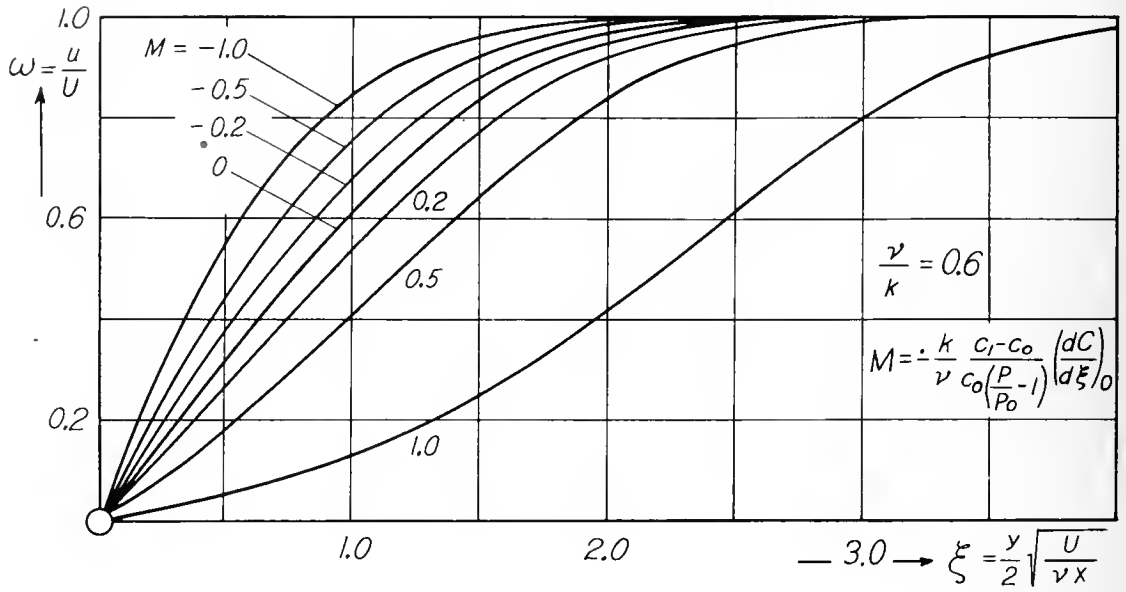


Figure 5.- Velocity field at diffusion with higher concentrations, where finite transverse velocities occur at the wall (see text for equations (18) to (20)).

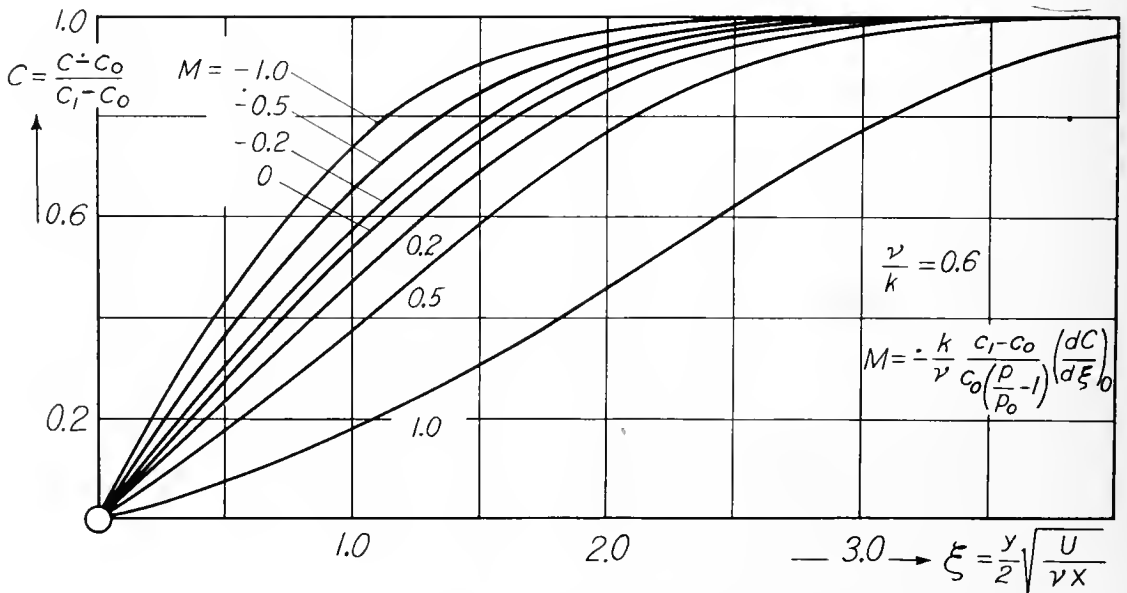


Figure 6.- Concentration distribution to figure 5.

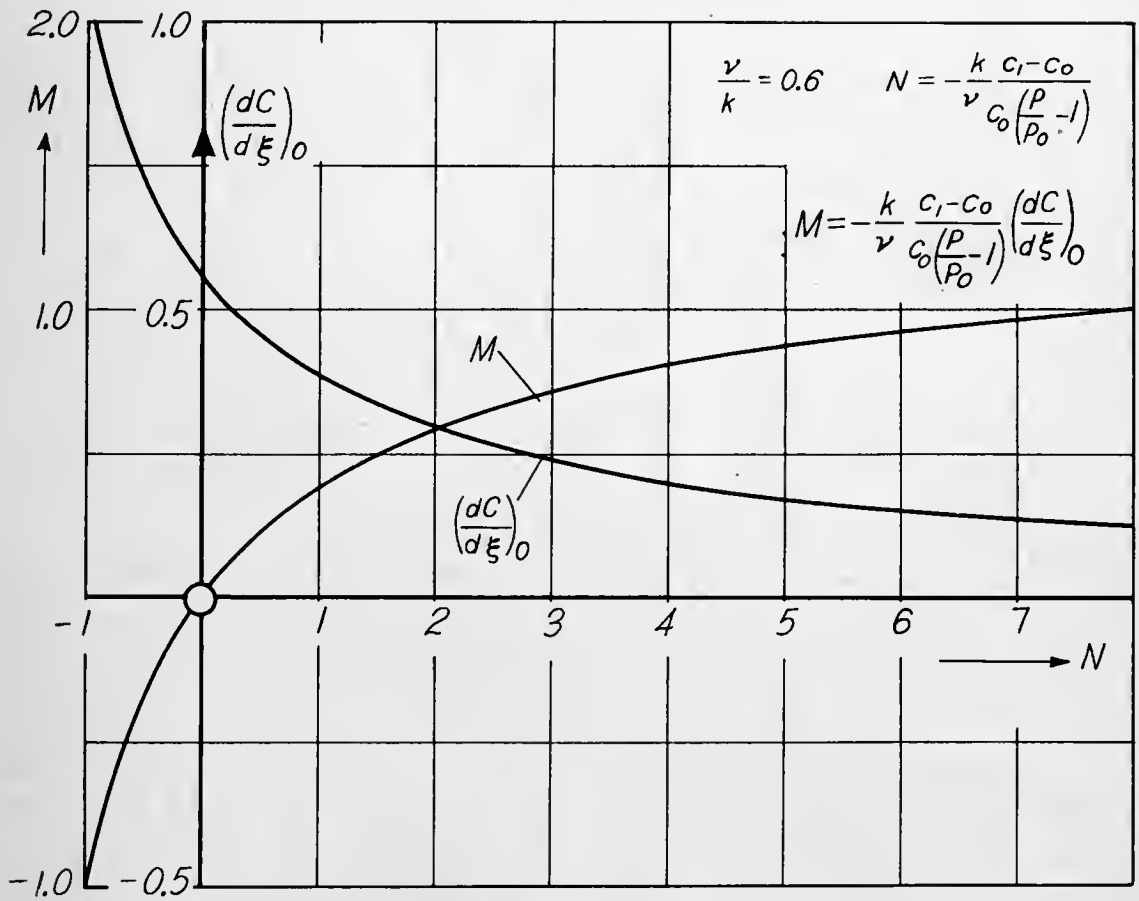


Figure 7.- Concentration gradient at the wall and quantity  $M$  plotted against  $N$  (see text for equations (19) and (20)).



Flow, Laminar

1.1.3.1



The Solution of the Laminar-Boundary-Layer Equation for the Flat Plate for Velocity and Temperature Fields for Variable Physical Properties and for the Diffusion Field at High Concentration.

By H. Schuh

NACA TM 1275  
May 1950

(Abstract on Reverse Side)

Heat Transfer, Aerodynamic

1.1.4.2



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May 1950

(Abstract on Reverse Side)

#### Abstract

The problem of finding a solution of the partial differential boundary-layer equations of motion and energy for the flow over a flat plate is changed to the problem of finding a solution of two integral equations. The two integral equations are solved by an iteration process which provides the velocity and temperature distributions across the boundary layer. The velocity and temperature distributions through the boundary layer on a flat plate situated in a very viscous fluid and in air are calculated by the iteration method. A solution is also given for the problem of the diffusion of a gas or a vapor from a flat plate placed in a flowing stream.

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